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Extended superspace, higher derivatives and $SL(2, Z)$ duality

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Abstract

We consider the low-energy effective action for the Coulomb phase of an $N = 2$ supersymmetric gauge theory with a rank one gauge group. The $N = 2$ superspace formalism is naturally invariant under an $SL(2, \mathbf{Z})$ group of duality transformations, regardless of the form of the action. The leading and next to leading terms in the long distance expansion of the action are given by the holomorphic prepotential and a real analytic function respectively. The latter is shown to be modular invariant with respect to $SL(2, \mathbf{Z})$.

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1. Introduction

In the last year and a half, there has been dramatic progress in our understanding of strongly coupled supersymmetric gauge theories in four dimensions. (See for example the recent review by Seiberg [1].) In particular, Seiberg and Witten [2] were able to determine the metric on the Coulomb branch of the moduli space of vacua of $N = 2$ supersymmetric $SU(2)$ gauge theories with various matter content.

The light degrees of freedom on the Coulomb branch of such theories constitute an $N = 2$ vector multiplet, i.e. an $N = 1$ vector multiplet and an $N = 1$ uncharged chiral multiplet. The method used in [2] and many subsequent papers was to study the terms in the low-energy effective action for these fields with at most two space-time derivatives or four fermions. These terms are determined by a holomorphic prepotential. An $SL(2, \mathbf{Z})$ group of duality transformations, acting linearly on the $N = 1$ chiral superfield and the first derivative of the prepotential and by electric-magnetic duality on the $N = 1$ vector field, plays an important role in determining the properties of the model.

However, the prepotential terms are just the leading terms in a systematic long-distance expansion of the low-energy effective action. The object of this paper is to study the exact expansion, in particular the next to leading terms. For simplicity, we will only consider the case of a rank one gauge group; the generalization to other groups should be straightforward.

It is convenient to work in a manifestly $N = 2$ supersymmetric formalism, and in section two we give a quick review of $N = 2$ superspace and in particular the $N = 2$ vector multiplet. In section three, we show that this formalism is naturally invariant under a group of duality transformations isomorphic to $SL(2, \mathbf{Z})$, regardless of the form of the action. In section four, we discuss the long-distance expansion of the low-energy effective action in $N = 1$, $N = 2$ and $N = 4$ supersymmetric theories. In the case of $N = 2$ supersymmetry, the leading and next to leading terms are given by the holomorphic prepotential and a real analytic function respectively. In section five, we show how these objects transform under $SL(2, \mathbf{Z})$. For the prepotential, we recover the results of [2] that its first derivative and the fundamental field transform in the defining representation of $SL(2, \mathbf{Z})$. The real analytic function is found to be modular invariant with respect to $SL(2, \mathbf{Z})$.

2. The $N=2$ superspace formalism

We will be using the formalism of Grimm, Sohnius and Wess [3]. The $N=2$ superspace has bosonic coordinates x^μ and fermionic coordinates $\theta^\alpha{}_i$ and $\bar\theta^{\dot\alpha i}$, where $\mu=0,\ldots,3$ is a space-time vector index, $\alpha=1,2$ and $\dot\alpha=\dot 1,\dot 2$ are Weyl and anti-Weyl spinor indices respectively, and $i=1,2$ indexes a doublet under the $SU(2)_R$ algebra which is part of the $N=2$ supersymmetry algebra.

Infinitesimal supersymmetry transformations are generated by $Q_\alpha{}^i$ and $\bar Q_{\dot\alpha i}$ defined by

$$\begin{aligned} Q_\alpha{}^i &= \frac{\partial}{\partial\theta^\alpha{}_i} - i(\sigma^\mu\bar\theta^i)_\alpha\partial_\mu \\ \bar Q_{\dot\alpha i} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} + i(\theta_i\sigma^\mu)_{\dot\alpha}\partial_\mu . \end{aligned} \tag{2.1}$$

They fulfill the algebra $\{Q_\alpha{}^i, Q_\beta{}^j\} = \{\bar Q_{\dot\alpha i}, \bar Q_{\dot\beta j}\} = 0$, $\{Q_\alpha{}^i, \bar Q_{\dot\beta j}\} = 2\sigma^\mu{}_{\alpha\dot\beta}\delta^i_j P_\mu$, where P_μ is the space-time translation operator. We will also have use for the super-covariant derivatives $D_\alpha{}^i$ and $\bar D_{\dot\alpha i}$ defined by

$$\begin{aligned} D_\alpha{}^i &= \frac{\partial}{\partial\theta^\alpha{}_i} + i(\sigma^\mu\bar\theta^i)_\alpha\partial_\mu \\ \bar D_{\dot\alpha i} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - i(\theta_i\sigma^\mu)_{\dot\alpha}\partial_\mu . \end{aligned} \tag{2.2}$$

They anti-commute with the supersymmetry generators (2.1) and fulfill the algebra

$$\begin{aligned} &\{D_\alpha{}^i, D_\beta{}^j\} = \{\bar D_{\dot\alpha i}, \bar D_{\dot\beta j}\} = 0 \\ &\{D_\alpha{}^i, \bar D_{\dot\beta j}\} = -2i\sigma^\mu{}_{\alpha\dot\beta}\delta^i_j\partial_\mu . \end{aligned} \tag{2.3}$$

Any superfield integrated over all of superspace with the measure $d^4\theta d^4\bar\theta$ transforms into a total space-time derivative under the supersymmery transformations generated by (2.1). The same applies to a chiral (anti-chiral) superfield, i.e. a superfield annihilated by $\bar D_{\dot\alpha i}$ $(D_\alpha{}^i)$, integrated over chiral (anti-chiral) superspace with the measure $d^4\theta$ $(d^4\bar\theta)$.

The $N=2$ vector multiplet may be described by a complex superfield A in the adjoint representation of the gauge group which fulfills a chirality constraint

$$\bar D_{\dot\alpha i} A = 0 \tag{2.4}$$

and a Bianchi identity constraint

$$D^{\alpha i} D_\alpha{}^j A = \bar D_{\dot\alpha}{}^i \bar D^{\dot\alpha j} \bar A . \tag{2.5}$$

In terms of $N = 1$ superfields, A contains a vector field strength W_α and an $N = 1$ chiral field Φ obeying the usual constraints $\bar{D}_{\dot\alpha} W_\alpha = 0$, $D^\alpha W_\alpha = \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha}$ and $\bar{D}_{\dot\alpha} \Phi = 0$. (Our conventions for $N = 1$ superspace are those of Wess and Bagger [4].) Decomposing further, we see that the component fields are a gauge potential A_μ, a complex scalar ϕ, an $SU(2)_R$ doublet $\lambda_\alpha{}^i$ of Weyl fermions and an $SU(2)_R$ triplet E^A, $A = 1, 2, 3$, of auxiliary fields.

In the abelian case, the constraints (2.4) and (2.5) were solved by Mezincescu [5] in terms of a real and symmetric but otherwise unconstrained superfield V_{ij} as

$$A = \bar{D}^4 D^{\alpha i} D_\alpha{}^j V_{ij}. \tag{2.6}$$

3. The duality transformations

An important fact about $N = 2$ supersymmetric gauge theories is that they in general have a moduli space of inequivalent vacuum states. We will be considering the Coulomb branch of this moduli space, where the gauge group is spontaneously broken down to its maximal abelian subgroup. For simplicity of notation, we will only consider the case of a rank one gauge group; the generalization to larger groups should be straightforward.

The light degrees of freedom at a generic point on the Coulomb branch constitute an $N = 2$ $U(1)$ vector multiplet A, and the physics is described at low energies by some effective action functional $\mathcal{S}[A, \bar{A}]$. (This low energy effective action is in general not renormalizable, although the underlying microscopic theory must be so.) The partition function of the model is then given by

$$\mathcal{Z} = \int \mathcal{D}V_{ij} \exp i\mathcal{S}[A, \bar{A}]. \tag{3.1}$$

We emphasize that the quantity A in this expression is not an independent field, but given in terms of the real but otherwise unconstrained field V_{ij} as in (2.6).

In this section, we will show that this formalism is naturally invariant under a group of duality transformations isomorphic to $SL(2, \mathbf{Z})$. These transformations are not symmetry transformations in the traditional sense, but rather relate different descriptions (i.e. different action functionals of different fields) of the same theory. The group $SL(2, \mathbf{Z})$ can be thought of as generated by two elements S and T with the relations

$$\begin{aligned} S^2 &= 1 \\ (ST)^3 &= 1, \end{aligned} \tag{3.2}$$

where 1 denotes the identity element.

We will first describe the action of the S-transformation on the model. To this end, we begin by rewriting the partition function (3.1) by letting A be a chiral but otherwise unconstrained field (i.e. not necessarily obeying the Bianchi constraint (2.5)). This constraint is then enforced by means of a real and symmetric Lagrange multiplier field $\tilde{V}_{ij}$. We thus write the partition function as

$$\mathcal{Z} = \int \mathcal{D}A\,\mathcal{D}\bar{A}\,\mathcal{D}\tilde{V}_{ij} \exp i \left(S[A,\bar{A}] + \int d^4x d^4\theta d^4\bar{\theta} \tilde{V}_{ij} (D^{\alpha i} D_\alpha{}^j A - \bar{D}_{\dot{\alpha}}{}^i \bar{D}^{\dot{\alpha} j} \bar{A}) \right). \quad (3.3)$$

Performing the path integral over $\tilde{V}_{ij}$ gives rise to a delta functional which enforces the constraint (2.5). The general solution to this constraint is paramerized by a real but otherwise unconstrained field V_{ij} as in (2.6), so the partition functions (3.1) and (3.3) are indeed equal. (The path-integral measures in these expressions are determined, up to an overall factor, by $N = 2$ supersymmetry.) The S-transformation now amounts to instead integrating out A (and its complex conjugate $\bar{A}$) from (3.3). The partition function is then given in the dual form

$$\mathcal{Z} = \int \mathcal{D}\tilde{V}_{ij} \exp i\tilde{\mathcal{S}}[\tilde{A}, \bar{\tilde{A}}], \quad (3.4)$$

where $\tilde{A} = \bar{D}^4 D^{\alpha i} D_\alpha{}^j \tilde{V}_{ij}$ and the dual action $\tilde{\mathcal{S}}[\tilde{A}, \bar{\tilde{A}}]$ is given by

$$\exp i\tilde{\mathcal{S}}[\tilde{A}, \bar{\tilde{A}}] = \int \mathcal{D}A\,\mathcal{D}\bar{A} \exp i \left(\mathcal{S}[A,\bar{A}] + \int d^4x d^4\theta A\tilde{A} + \text{c.c.} \right). \quad (3.5)$$

Here we have used that the factor $d^4\bar{\theta}$ in the superspace integration measure can be replaced by $\bar{D}^4$ and analogously for the complex conjugate. We see that the dual expression (3.4) for the partition function is of the same form as the original expression (3.1), but with a new action $\tilde{\mathcal{S}}$ which is a functional of the new variable $\tilde{V}_{ij}$ (through the quantities $\tilde{A}$ and $\bar{\tilde{A}}$). The relationship (3.5) states that the S-transformation acts as a Fourier transformation in field space on the exponentiated action of the model.

We now turn to the T-transformation, which simply consists of adding the terms

$$\int d^4\theta \frac{1}{2} AA + \int d^4\bar{\theta} \frac{1}{2} \bar{A}\bar{A} = \frac{1}{32\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \quad (3.6)$$

to the Lagrangian of the model. This amounts to shifting the θ-angle by 2π and thus gives an equivalent description of the theory. The new action is still a function of the same independent field V_{ij} (through the quantities A and $\bar{A}$).

To establish $SL(2,\mathbf{Z})$ invariance, we must check the relations (3.2). Two consecutive S-transformations transform an action functional $\mathcal{S}$ first into $\tilde{\mathcal{S}}$ given by (3.5), and then into $\tilde{\tilde{\mathcal{S}}}$ given by

$$\begin{aligned}\exp i\tilde{\tilde{\mathcal{S}}}[\tilde{\tilde{A}},\bar{\tilde{\tilde{A}}}] &= \int \mathcal{D}\tilde{A}\,\mathcal{D}\bar{\tilde{A}}\,\mathcal{D}A\,\mathcal{D}\bar{A}\exp i\left(\mathcal{S}[A,\bar{A}] + \int d^4x d^4\theta (A\tilde{A} + \tilde{A}\tilde{\tilde{A}}) + \text{c.c.}\right)\\ &= \exp i\mathcal{S}[-\tilde{\tilde{A}},-\bar{\tilde{\tilde{A}}}].\end{aligned} \tag{3.7}$$

This shows that $S^2 = 1$ on the action, up to the trivial change of variable $A \to -A$. Furthermore, a T-transformation followed by an S-transformation transform an action functional $\mathcal{S}$ into $\mathcal{S}_1$ given by

$$\exp i\mathcal{S}_1[A_1,\bar{A}_1] = \int \mathcal{D}A\mathcal{D}\bar{A}\exp i\left(\mathcal{S}[A,\bar{A}] + \int d^4x d^4\theta (\frac{1}{2}AA + AA_1) + \text{c.c.}\right). \tag{3.8}$$

If we now perform three consecutive ST-transformations, an action functional $\mathcal{S}$ is transformed into $\mathcal{S}_3$ given by

$$\begin{aligned}\exp i\mathcal{S}_3[A_3,\bar{A}_3] &= \int \mathcal{D}A_2\,\mathcal{D}\bar{A}_2\,\mathcal{D}A_1\,\mathcal{D}\bar{A}_1\,\mathcal{D}A\,\mathcal{D}\bar{A}\exp i\Big(\mathcal{S}[A,\bar{A}]\\ &\quad + \int d^4x d^4\theta (\frac{1}{2}AA + AA_1 + \frac{1}{2}A_1A_1 + A_1A_2 + \frac{1}{2}A_2A_2 + A_2A_3) + \text{c.c}\Big)\\ &= \exp i\mathcal{S}[A_3,\bar{A}_3].\end{aligned} \tag{3.9}$$

Thus $(ST)^3 = 1$ on the action. The relations (3.2) being fulfilled, the group generated by the S- and T-transformations is indeed isomorphic to $SL(2,\mathbf{Z})$.

4. The long-distance expansion

The exact effective action $\mathcal{S}[A,\bar{A}]$ obtained by integrating out all massive degrees of freedom from the microscopic theory is in general an intractable non-local expression. To be able to extract useful information about for example a scattering process, we can expand this expression in powers of the momentum scale of the external particles divided by the characteristic scale of the theory and consider the leading terms. Since a space-time derivative in the action corresponds to a power of space-time momentum, this procedure roughly amounts to keeping terms with up to some maximal number of space-time derivatives in the action.

We thus introduce an 'order in derivatives' n to the different objects in our theory as follows: We define n by specifying that the $N=2$ vector field A has $n=0$ and a super covariant derivative $D_\alpha{}^i$ has $n=\frac{1}{2}$. (n is always invariant under complex conjugation.) From the anti-commutators of covariant derivatives (2.3), it then follows that $n=1$ for a space-time derivative ∂_μ as we wanted. The space-time integration measure d^4x must have $n=-4$. From the explicit form of the derivatives (2.2), we see that the fermionic coordinates $\theta^\alpha{}_i$ have $n=-\frac{1}{2}$, and the rules of Grassmann integration then give that $n=2$ for the integration measure $d^4\theta$. Finally, we see that the delta functions $\delta^4(x)$ and $\delta^4(\theta)$ have $n=4$ and $n=-2$ respectively.

Decomposing A into an $N=1$ vector field W_α and an $N=1$ chiral field Φ, we see that these have $n=\frac{1}{2}$ and $n=0$ respectively. The integration measure $d^2\theta$ has $n=1$. For the component fields, we get $n=0$ for the gauge potential A_μ and the scalar field ϕ, $n=\frac{1}{2}$ for the spinor fields $\lambda_\alpha{}^i$, and $n=1$ for the auxiliary fields E^A. In particular, this means that the canonical kinetic terms for these fields (i.e. the terms in the free Lagrangian $i\int d^4\theta AA+\text{c.c.}$) will all have $n=2$. This, together with the requirement that a space-time derivative has $n=1$, could also be taken as a (perhaps more intuitive) definition of n. (We caution the reader not to confuse the order n with the canonical dimension d. For example, $d=1$ for the scalar field ϕ and the gauge potential A_μ, $d=\frac{3}{2}$ for the spinor fields $\lambda_\alpha{}^i$ and $d=2$ for the auxiliary fields E^A.)

In $N=1$ supersymmetry, it is possible to write terms in the Lagrangian of all integer orders n. The lowest order term in that case is the order $n=1$ term

$$\int d^2\theta f(\Phi)+\text{c.c.}, \tag{4.1}$$

where the superpotential f is an arbitrary holomorphic function. At order $n=2$ we have the terms

$$\int d^2\theta d^2\bar{\theta} K(\Phi,\bar{\Phi}) \tag{4.2}$$

and

$$\int d^2\theta \tau(\Phi) W^\alpha W_\alpha+\text{c.c.}, \tag{4.3}$$

where the Kähler potential K and the gauge coupling τ are arbitrary real analytic and holomorphic functions respectively. The number of possible terms then increases rapidly with the order n.

In $N=2$ supersymmetry, only terms of even order n are possible. At order $n=2$ we have

$$\int d^4\theta \mathcal{F}(A) + \text{c.c.} \tag{4.4}$$

for an arbitrary holomorphic prepotential $\mathcal{F}$ [6]. In terms of $N=1$ superfields, this term may be written as

$$\int d^2\theta d^2\bar{\theta}\bar{\Phi}\mathcal{F}'(\Phi) + \int d^2\theta \frac{1}{2}\mathcal{F}''(\Phi)W^\alpha W_\alpha + \text{c.c.}, \tag{4.5}$$

where a prime on a function denotes differentiation with respect to its argument. At order $n=4$ we have

$$\int d^4\theta d^4\bar{\theta}\mathcal{K}(A,\bar{A}) \tag{4.6}$$

for an arbitrary real analytic function $\mathcal{K}$. In $N=1$ superspace this reads

$$\begin{aligned}\int d^2\theta d^2\bar{\theta}\Big(\mathcal{K}_{\phi\bar{\phi}}(\Phi,\bar{\Phi})\big(&D^\alpha D_\alpha\Phi\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\bar{\Phi} + 2\bar{D}_{\dot{\alpha}}D^\alpha\Phi D_\alpha\bar{D}^{\dot{\alpha}}\bar{\Phi} + 4D^\alpha W_\alpha\bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \\ &- 4D^{(\alpha}W^{\beta)}D_{(\alpha}W_{\beta)} - 4\bar{D}_{(\dot{\alpha}}\bar{W}_{\dot{\beta})}\bar{D}^{(\dot{\alpha}}\bar{W}^{\dot{\beta})} \\ &- 2D^\alpha D_\alpha(W^\beta W_\beta) - 2\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}(\bar{W}_{\dot{\beta}}\bar{W}^{\dot{\beta}})\big) \\ - 2\mathcal{K}_{\phi\phi\bar{\phi}}(\Phi,\bar{\Phi})W^\alpha W_\alpha D^\beta D_\beta\Phi &- 2K_{\phi\bar{\phi}\bar{\phi}}(\Phi,\bar{\Phi})\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}^{\dot{\beta}}\bar{\Phi} \\ + K_{\phi\phi\bar{\phi}\bar{\phi}}(\Phi,\bar{\Phi})\left(-8W^\alpha D_\alpha\Phi\bar{W}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\bar{\Phi} + 4W^\alpha W_\alpha\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right)\Big)&.\end{aligned} \tag{4.7}$$

(The subscripts on the function $\mathcal{K}$ denote derivatives with respect to its arguments.) The expression (4.7) is unique only up to total space-time derivatives or terms that vanish because $D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$. The latter ambiguity makes the duality transformations less straightforward in the $N=1$ superspace formalism and is one of the reasons that we prefer to work in $N=2$ superspace.

At order $n=4$ a new phenomenon appears: If we expand out the Lagrangian (4.6) or (4.7) in component fields, we will see that it contains a term proportional to $\mathcal{K}_{\phi\bar{\phi}}(\phi,\bar{\phi})\partial_\mu E^A\partial^\mu E_A$. This means that the equations of motion for the auxiliary fields E^A will not be algebraic (unless $\mathcal{K}$ is a sum of a holomorphic and an anti-holomorphic term, in which case the action vanishes). The auxiliary fields are still non-propagating, though, because they may be solved for in the equations of motion as an infinite series of terms in the other fields of all orders n. (We recall that the auxiliary fields are 'nominally' of order $n=1$.) We remark that if we require only $N=1$ supersymmetry it is possible to find order $n=4$ terms in the Lagrangian such that the auxiliary fields have algebraic equations of motion, and this may be possible for $N=2$ supersymmetry as well if we go to higher orders n. However, the solution for the auxiliary fields to such equations of motion would still be of the same form, i.e. an infinite series in the dynamical fields containing terms of all orders n, so there is no reason to impose this extra requirement.

Finally, we will briefly consider the case of $N=4$ supersymmetry. This is less straightforward, since no off-shell formulation of $N=4$ supersymmetry is known. On-shell, an $N=4$ vector multiplet can be decomposed as an $N=1$ vector multiplet W_α and three $N=1$ chiral multiplets Φ_a, $a=1,2,3$. The unique order $n=2$ Lagrangian is

$$2\tau \int d^2\theta d^2\bar{\theta} \bar{\Phi}^a \Phi_a + \tau \int d^2\theta W^\alpha W_\alpha + \text{c.c.}, \tag{4.8}$$

where $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. ($\theta$ and g are the theta-angle and the gauge coupling constant respectively.) Putting the auxiliary fields to zero by their equations of motion, the action is invariant under the $SU(4)_R$ which is part of the $N=4$ supersymmetry algebra. Only an $SU(3) \times U(1)_R$ subgroup is manifest in (4.8), though, with W_α and Φ_a transforming in the $\mathbf{1}_3$ and $\mathbf{3}_2$ representations respectively. To construct an order $n=4$ term, we first consider the case when $\Phi_2 = \Phi_3 = 0$ (assuming that this limit is smooth). The remaining fields W_α and $\Phi = \Phi_1$ should then constitute an $N=2$ vector multiplet, and the Lagrangian must be of the form (4.7). Furthermore, invariance under $U(1)_R$ requires $\mathcal{K}$ in (4.7) to be a function of $\Phi\bar{\Phi}$ only. We now reinstate Φ_2 and Φ_3 such that the resulting Lagrangian is manifestly invariant also under $SU(3)$. Solving the equations of motion for the auxiliary fields and substituting back in the Lagrangian will only produce terms of order $n \geq 6$, so if we work at order $n=4$ we can put them to zero. Finally we must check that the remaining Lagrangian is really invariant under $SU(4)$. This turns out to require that $\mathcal{K}(\Phi, \bar{\Phi}) = k\Phi\bar{\Phi}$ for some constant k. Inserting this in (4.7) and reinstating Φ_2 and Φ_3 gives the unique term

$$k \int d^2\theta d^2\bar{\theta} (2D^\alpha D_\alpha \Phi_a \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\Phi}^a - 8D^\alpha W_\alpha \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}) \tag{4.9}$$

up to a total space-time derivative and terms proportional to $D^\alpha W_\alpha - \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}$. Since the term (4.9) is bilinear in the fields, we may in fact put the auxiliary fields to zero to all orders n by their equations of motion. It is conceivable that terms in the Lagrangian of higher order n depending on some arbitrary functions, as in the case of $N=1$ or $N=2$ supersymmetry, may be constructed. However, such functions would have to be real analytic rather than holomorphic, since the only $SU(4)$ invariant combination of the scalar component fields, namely $\phi_a \bar{\phi}^a$, is real.

5. Transformation properties of the effective action

To determine the transformation properties of the different terms in the effective action $\mathcal{S}$ under $SL(2,\mathbf{Z})$ duality, it is convenient to decompose it as

$$\mathcal{S}[A,\bar{A}] = \widehat{\mathcal{S}}[A,\bar{A}] + \int d^4x d^4\theta \mathcal{F}(A) + \text{c.c.}, \tag{5.1}$$

where the first term contains all contributions to the action of order $n \geq 0$. (In this section, the order n will always refer to terms in the action as opposed to the Lagrangian, i.e. including the $n = -4$ contribution of the space-time integration measure. The second term in (5.1) thus has $n = -2$.) Inserting (5.1) in the formula (3.5) for the dual action after an S-transformation and decomposing this in analogy to (5.1), we get

$$\begin{aligned}
\exp i\tilde{\mathcal{S}}[\tilde{A},\bar{\tilde{A}}] &= \exp i\left(\widehat{\tilde{\mathcal{S}}}[\tilde{A},\bar{\tilde{A}}] + \int d^4x d^4\theta \tilde{\mathcal{F}}(\tilde{A}) + \text{c.c.}\right) \\
&= \int \mathcal{D}A\mathcal{D}\bar{A} \exp i\left(\widehat{\mathcal{S}}[A,\bar{A}] + \int d^4x d^4\theta (\mathcal{F}(A) + A\tilde{A}) + \text{c.c.}\right) \\
&= \int \mathcal{D}A\mathcal{D}\bar{A} \exp i\left(\widehat{\mathcal{S}}[-i\frac{\delta}{\delta\tilde{A}}, -i\frac{\delta}{\delta\bar{\tilde{A}}}] + \int d^4x d^4\theta (\mathcal{F}(A) + A\tilde{A}) + \text{c.c.}\right) \\
&= \exp i\widehat{\mathcal{S}}[-i\frac{\delta}{\delta\tilde{A}}, -i\frac{\delta}{\delta\bar{\tilde{A}}}] \exp i\left(\int d^4x d^4\theta \tilde{\mathcal{F}}(\tilde{A}) + \text{c.c.}\right),
\end{aligned} \tag{5.2}$$

where the dual prepotential $\tilde{\mathcal{F}}$ is given by

$$\exp i \int d^4x d^4\theta \tilde{\mathcal{F}}(\tilde{A}) = \int \mathcal{D}A \exp i \int d^4x d^4\theta (\mathcal{F}(A) + A\tilde{A}) \tag{5.3}$$

and the functional derivative is defined by

$$\frac{\delta A(x',\theta',\bar{\theta}')}{\delta A(x,\theta,\bar{\theta})} = \bar{D}'^4 \left(\delta^4(x-x')\delta^4(\theta-\theta')\delta^4(\bar{\theta}-\bar{\theta}')\right). \tag{5.4}$$

The four $\bar{D}'$ operators on the right-hand side of (5.4) arise because A obeys the chirality constraint (2.4). We note that the operator $\frac{\delta}{\delta A}$ is of order $n = 2$.

Although in general not a Gaussian, the path integral in (5.3) is not difficult to evaluate. We require that the function A_D, defined by

$$A_D = \mathcal{F}'(A), \tag{5.5}$$

assumes every value exactly once. It then follows that, given $\tilde{A}$, the integrand in (5.3) has a unique stationary point given by the equation $\mathcal{F}'(A) + \tilde{A} = 0$, i.e.

$$\tilde{A} = -A_D. \tag{5.6}$$

We denote the corresponding value of A as $\tilde{A}_D$ so that

$$\mathcal{F}'(\tilde{A}_D) + \tilde{A} = 0. \tag{5.7}$$

We now claim that the path integral (5.3) is given by the value of the integrand at this point, i.e. $\exp i \int d^4x d^4\theta \tilde{\mathcal{F}}(\tilde{A}) = \exp i \int d^4x d^4\theta (\mathcal{F}(\tilde{A}_D) + \tilde{A}_D \tilde{A})$. The dual prepotential is thus given by

$$\tilde{\mathcal{F}}(\tilde{A}) = \mathcal{F}(\tilde{A}_D) + \tilde{A}_D \tilde{A}. \tag{5.8}$$

The behavior of the integrand in (5.3), apart from its value at the stationary point, is of no consequence, since we can change it to any fiducial function by a holomorphic change of variables, $A \to y(A)$. It is not difficult to check that the path integration measure $\mathcal{D}A$ is invariant under such a transformation, the bosonic and fermionic contributions to the Jacobian canceling because of supersymmetry. The path integral (5.3) thus equals the value of the integrand at the stationary point (times a normalization constant which we can put to one).

Differentiating (5.8) with respect to $\tilde{A}$ and using (5.7), we get

$$\tilde{A}_D = \tilde{\mathcal{F}}'(\tilde{A}), \tag{5.9}$$

which is the dual counterpart of (5.5). Inserting this back in (5.7) we get $\mathcal{F}'(\tilde{\mathcal{F}}'(\tilde{A})) = -\tilde{A}$. An analogous formula appeared in the $N = 1$ superspace formalism in [2]. The relationship, at the stationary point, between the variables before and after dualization can be summarized as

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \to \begin{pmatrix} \tilde{A}_D \\ \tilde{A} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix}. \tag{5.10}$$

We now return our attention to the expression (5.2) for the dual action. Consider a term in $\widehat{\mathcal{S}}[A, \bar{A}]$ of some definite order n. Replacing a power of A by $-i\frac{\delta}{\delta \tilde{A}}$ and letting it act on an object of order n' would, by (5.4), produce a result of order $n + n' + 2$. But $n' \geq -2$ with equality only for the dual prepotential $\int d^4x d^4\theta \tilde{\mathcal{F}}(\tilde{A}) + \text{c.c.}$, so we see that a term of order n in the action will only affect the terms of equal or higher order in the dual action. The terms of equal order arise when we let all functional derivatives act on the exponentiated dual prepotential $\exp i \int d^4x d^4\theta \tilde{\mathcal{F}}(\tilde{A}) + \text{c.c.}$ so that $-i\frac{\delta}{\delta \tilde{A}}$ can be replaced by $\tilde{\mathcal{F}}'(\tilde{A}) = \tilde{A}_D$. At the stationary point, we can furthermore use (5.10) to replace the latter expression by A.

In particular, consider the terms of order $n = 0$ in the dual action (corresponding to the terms of order $n = 4$ in the Lagrangian) given by a real analytic function $\tilde{\mathcal{K}}$ as we discussed in the previous section. This function is completely determined by the terms of order $n \leq 0$ in the original action, i.e. by the prepotential $\mathcal{F}$ and the real analytic function $\mathcal{K}$. From the discussion in the previous paragraph, it follows that

$$\tilde{\mathcal{K}}(\tilde{A}, \bar{\tilde{A}}) = \mathcal{K}(A, \bar{A}), \tag{5.11}$$

where by A we understand its value at the stationary point of the integrand in (5.3) as a function of $\tilde{A}$.

The T-transformation obviously acts as $\mathcal{F}(A) \rightarrow \mathcal{F}(A) + \frac{1}{2}AA$, leaving $\widehat{\mathcal{S}}[A, \bar{A}]$ and in particular $\mathcal{K}(A, \bar{A})$ invariant. This can also be stated as

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix}. \tag{5.12}$$

As in [2], the interpretation of (5.10) and (5.12) is that $\begin{pmatrix} A_D \\ A \end{pmatrix}$ constitutes a section of an $SL(2, \mathbf{Z})$ bundle over the moduli space of vacua. Our result (5.11) concerning the terms in the action of order $n = 0$ and the corresponding statement for the T-transformation can then be summarized by saying that the real analytic function $\mathcal{K}$ is a modular function with respect to $SL(2, \mathbf{Z})$. The method of this section could in principle be applied to determine how the terms in the action of any given order n behave under duality transformations. (We have already seen that their transforms are determined by terms of equal or lower order in the original action.)

Much of the recent progress in supersymmetric gauge theories relies on the holomorphicity of the superpotential in $N = 1$ theories or the prepotential in $N = 2$ theories. When combined with a knowledge of singularity structure and/or asymptotic behavior, this is often enough to completely determine these functions. Unfortunately, we have seen that the higher order contributions to the effective action are given by real analytic rather than holomorphic objects, so they cannot be determined by this method. However, in the case of $N = 2$ supersymmetry, the Coulomb branch of the moduli space of vacua can be identified with a certain family of Riemann surfaces, and the prepotential is then closely related to the periods of a certain one-form. It is conceivable that also the higher order terms in the effective action, such as those determined by the function $\mathcal{K}(A, \bar{A})$ that we have discussed, have a geometric interpretation. We hope that the results of the present paper may be helpful in clarifying this structure.

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